Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

REVIEWS OF BOOKS.

An Essay on the Foundations of Geometry. By BERTRAND A. W. RUSSELL, M.A., Fellow of Trinity College, Cambridge. Cambridge, The University Press, 1897.—pp. xvi, 201.

Geometry possessed much of interest for philosophers in the last century when a belief in the perfection and completeness of the science held sway. It has much more interest for them at the present time, for Geometry is now known to be a science broader in its range and more varied in its methods than the philosophers of a hundred years ago had imagined. A great deal more has been done on the Continent than in England and America in the important and difficult work of examining the foundations of a science which has received wonderful and extensive additions especially in the last three decades. Accordingly, the appearance of an English work dealing with the Foundations of Geometry may be regarded as an event worthy of note.

A complete discussion on the foundations of Geometry would include what students of Metaphysics and Logic, what the psychologist and the physicist, as well as what the mathematician would have to say. In this essay, Mr. Russell confines himself solely to the *logical* questions and problems which arise in connection with his subject; he leaves to others the treatment of Geometry from the point of view of psychology and metaphysics. He has given us a volume which is valuable both to the mathematician and to the philosopher. The chief significance of this work, which the reviewer will attempt to summarize rather than to criticise, lies in the fact that old questions of fundamental importance are raised anew in the light of modern advances in mathematical research; and at a time, too, when the problems dealt with are claiming especial interest and attention. The book is written in a lucid and attractive style, and its matter is well arranged and clearly presented. Its most important feature is that it contains the first presentation of projective Geometry yet made to philosophers. Hitherto, the latter have not taken account of projective Geometry, because they have not perceived the distinction between it and metrical Geometry.

The essay is divided into four chapters. Chapter I. gives a short history of the geometries which introduce ideas different from those of Euclid; and Chapter II. contains a critical account of some previous philosophical theories of Geometry. The way is thus prepared for Chapter III., which is constructive and is the principal part of the book. It consists of two sections which are devoted respectively to the axioms of projective Geometry and metrical Geometry. Some questions of a less geometrical and a more general philosophical nature which arise in Chapter III. are discussed in the final chapter.

Mr. Russell explains carefully the sense in which he uses the word *a priori*. He defines *a priori* knowledge as consisting of "the postulates which are necessary to make knowledge possible at all and of all that can be deduced from these postulates" (pp. 2–5, 60). His test of apriority is purely logical. The term *a priori* is used without any psychological implication ; the discussion of what is subjective is regarded as a part of Psychology, and the question of the relation of the *a priori* to the subjective as belonging to Metaphysics. The history of Metageometry aims to set forth the leading mathematical principles of successive periods in the development of Geometry, and to furnish the reader merely with the technical knowledge sufficient to enable him to consider the logic of the subject. On this account the sketch is brief, and does not refer to the work of some of the great modern geometers. Probably most students of philosophy will require the assistance of a mathematical friend in reading some portions of the essay.

The history is divided into three periods which differ from one another in their philosophical and mathematical aims and methods. In the first period, which is marked by the names of Gauss, Lobatchewsky and Bolyai, logically consistent Geometries were obtained by denying Euclid's axiom of parallels. This axiom was, therefore, shown to be logically independent of the others and essential to the Euclidean system of Geometry. The second period was under the guidance of a philosophical spirit which "aimed at no less than a logical analysis of all the essential axioms of Geometry." Riemann and Helmholtz were the chief investigators of this period, and the two fundamental conceptions used were that of *a manifold* and that of *the measure of curvature.* "Conceptions of magnitude, according to Riemann, are possible there only, where we have a general conception, capable of various determinations. The various determinations of such a conception together form a *manifold*, which is continuous or discrete, according as the passage from one determination to another is continuous or discrete" (p. 15). Space is defined as a species of manifold. This definition assumes that space can be regarded as a quantity, but leaves obscure the ground for regarding space as a system of magnitudes at all. The conception 'measure of curvature,' which was first applied by Gauss to surfaces, was extended by Riemann to a manifold of n dimensions. It is often forgotten that in the latter connection the measure of curvature "is a purely analytical expression which has only a symbolic affinity to ordinary curvature." The essay states that, "Riemann, in spite of his desire to prove that all the axioms can be dispensed with, has nevertheless, in his mathematical work, retained three fundamental axioms, namely, free mobility, the finite integral number of dimensions, and the axiom that two points have a unique relation, namely, distance" (p. 22).

In the third period, space is dealt with by descriptive rather than quantitative methods. Projective Geometry deals only with quality—for which reason it is called descriptive—and cannot distinguish between two figures which are qualitatively alike. Operations with quantity are employed in projective Geometry, but the quantities used are not spatial magnitudes. The coördinates used in projective Geometry "are not coördinates in the ordinary metrical sense, *i. e.*, the numerical measures of certain spatial magnitudes. On the contrary, they are a set of numbers, arbitrarily but systematically assigned to different points, like the numbers of houses in a street" (p. 119). But for the brevity of the alphabet they might as well be letters. The usual method of defining projective coördinates involves the notion of distance measured in the ordinary way. Klein has shown, however, that by means of a certain construction these coördinates can be defined by purely descriptive properties. Cayley suggested a particular function of the projective coördinates of two points as a new definition of the distance between them on a Euclidean plane. If the projective coördinates be chosen in a particular way, the value of this function is the same as that of the ordinary distance between the two points; if the projective coördinates be chosen in certain other ways, this function will give the distances between the two points in the various non-Euclidean Geometries of three dimensions. It follows that we obtain the formulæ of the several Geometries according as we choose our projective coördinates, that to each proposition in one Geometry there corresponds a proposition in each of the other Geometries, and that if there is a contradiction in one of them, there is a corresponding contradiction in each of the others. Geometry thus appears to depend upon the definition of distance applied on a Euclidean plane. This definition is arbitrary, and therefore some look upon geometric axioms as mere conventions. Mr. Russell, however, argues that projective properties are inadequate to express metrical properties, because the former have no metrical presupposition. The reduction of metrical to projective properties is merely a technical device, and the values obtained for the function employed in the projective definition of distance should not be confused with distance in the ordinary sense. This reduction depends, with the exception of the case of hyperbolic space, on the use of imaginary figures, and hence the author takes up Cayley's challenge to philosophers made at the meeting of the British Association in 1883, concerning the right of philosophy to ignore the use of imaginaries which play such a large and essential part in modern Analysis and Geometry. The first chapter is summed up thus: "We have seen how the philosophical motive, at first predominant, has been gradually extruded by the purely mathematical and technical spirit of most recent Geometers. At first, to discredit the Transcendental Æsthetic seemed, to Metageometers, as important as to advance their science; but from the works of Cayley, Klein or Lie, no reader could gather that Kant had ever lived. We have also seen, however, that as the interest *in* philosophy waned, the interest *for* philosophy increased: as the mathematical results shook themselves free from philosophical controversies, they assumed gradually a stable form, from which further development, we may reasonably hope, will take the form of growth, rather than transformation" (p. 50).

The modern representative theories of Geometry, beginning with that of Kant, are criticised in the second chapter. Metageometry has destroyed Kant's argument that since Geometry is known to have apodeictic certainty, therefore space must be *a priori* and subjective. But Kant argues conversely that, since it follows from grounds independent of Geometry that space must be *a priori*, therefore, Geometry must have apodeictic certainty. This argument is attacked on the philosophic side. The discussion on Kant's treatment of space involves a review of the arguments against his theory of synthetic and analytic judgments. The conclusion regarding Kant's argument for the apriority of space is that its "*logical* scope extends, not to Euclidean space, but only to any form of externality which could exist intuitively, and permit knowledge in beings with our laws of thought, of a world of diverse but interrelated things" (p. 62).

Riemann's definition of a manifold is obscure. Moreover, a qualitative basis must be implied in the definition of space as a collection of magnitudes, but none of the essential properties of space can emerge from the conception of space as a magnitude. Hence dissent is taken from the disjunction which underlies Riemann's philosophy of space: namely, either the axioms must be consequences of general conceptions of magnitude, or else they can be proved only by experience (p. 65).

It is shown that Helmholtz did not rigidly apply his criterion for the empirical. He also asserted that we *can* imagine non-Euclidean spaces, things which can be described in conceptual terms being regarded as imaginable; but "this is not the sense required for argumentation in this case." The most important question in Helmholtz's theory is the relation of Geometry to Mechanics involved in his appeal to rigid bodies. He seems to think that we infer the homogeneity of space from our experience of rigid bodies. But "to make Geometry await the perfection of Physics, is to make Physics, which depends throughout on Geometry, forever impossible. As well might we leave the formation of numbers until we had counted the houses in Piccadilly" (p. 81). The author, however, thinks that some sort of matter is essential to Geometry. "This geometrical matter is a more abstract and wholly different matter from that of Dynamics," and possesses nothing but spatial adjectives. Erdmann, the strongest philosophical defender of the theories of Riemann and Helmholtz, has ultra-empirical views. His theory is not applicable to projective Geometry. Following Riemann, he subsumes space as a whole under a general conception of magnitude. The criticism on this involves a discussion on the logical nature of judgments of magnitude. The latter are judgments of comparison, and quality is supposed identical in the object whose magnitude is stated and in the unit with which it is compared. Now there is a difference of quality in the various spaces, and therefore there is no qualitatively similar unit in the three kinds of space. Moreover, two different spaces cannot coexist in the same world and hence comparison is impossible. "The fact seems to be that Erdmann, in his admiration for Riemann and Helmholtz, has fallen in with their mathematical bias, and assumed, as mathematicians naturally tend to assume, that quantity is everywhere and always applicable and adequate, and can deal with more than the mere comparison of things whose qualities are already known as similar" (pp. 85 f). It is shown also that Erdmann's conclusions regarding the principles of Geometry "are largely invalidated by the diversity and inadequacy of his tests of the *a priori.*"

Lotze discusses the various meanings logically assignable to the proposition that other spaces than Euclid's are possible, and endeavors to show that the logic of the non-Euclideans is faulty. Mr. Russell holds that it is the philosophical bearing of Metageometry alone which constitutes its real importance. Metageometry has suggested the proof of the possibility of the existence of non-Euclidean spaces which conform to certain logical conditions that may be summed up in the relativity of position. Hence the truth of Euclid is determined by experience. "They (non-Euclidean spaces) throw light on the nature of the grounds for Euclid, rather than on the actual conformation of space" (p. 98). Lotze's attack on non-Euclidean logic is weak. Some of his arguments in this connection fail, because he is mistaken in his elementary mathematics; other arguments are worthless because he is mistaken as to what Metageometry really is. Lotze obtained his ideas concerning Metageomety from Helmholtz's attempts at popular explanation of the subject. It is unfortunate that philosophers should regard Helmholtz as the typical exponent of Metageometry, especially "after the very material advances brought about by the projective treatment of the subject. It is also unfortunate that his somewhat careless attempts to popularize mathematical results have so often been disposed of, without due attention to his more technical and solid contributions. Thus his romances about Flatland and Sphereland—at best only fairy-tale analogies of doubtful value—have been attacked as if they formed an essential feature of Metageometry" (p. 101). It is the opinion of the essayist that the recent speculations in France on the foundations of Geometry have added but little to geometrical philosophy. After the history of Metageometry and the philosophical criticisms described above, it is stated that "the logical validity of Metageometry, and the mathematical possibility of three-dimensional non-Euclidean spaces, will therefore be regarded, throughout the remainder of the work, as sufficiently established" (p. 109).

Chapter III. deals with the axioms of projective and metrical Geometry respectively. Since pure projective Geometry is not concerned with magnitude, and Euclidean and non-Euclidean spaces differ merely in their metrical properties, we may expect that in projective Geometry we can find the properties which are necessary and common to all possible spaces. The chapter begins with the mathematical explanations necessary to show how the "qualitative science of abstract externality, which is projective Geometry," is built up. Qualitative definitions of anharmonic ratio and harmonic range, the method of assigning projective coördinates to different points on a line, and the fundamental operations of projective Geometry are set forth; and the general principle of projective equivalence is stated and explained. "Two sets of points or of lines, which have the same anharmonic ratio, are treated by projective Geometry as equivalent: this qualitative equivalence replaces the quantitative equality of metrical Geometry" (p. 123). It is shown that the only reason within projective Geometry for not regarding projective figures as actually identical, is the intuitive perception of difference of position. This is "the essence of the notion of a form of externality, which notion forms the subject-matter of projective Geometry." The essential axioms on which the reasoning of projective Geometry is based are stated and proved to be necessary to any form of externality, the latter term being defined as the bare possibility of diversity among interrelated things. It is found that the most obvious prerequisite of absolute qualitative equivalence is the homogeneity of space, and hence that all position is relative. Homogeneity and relativity of position are shown to belong of necessity to any form of externality, and are, accordingly, *a priori* properties of all possible spaces. Next is discussed the principle that any form of externality must have a finite integral number of dimensions. Finally, it is proved that any two positions must have a relation independent of any reference to other positions. In spatial terms this relation is the straight line joining the two points. The difficulties which appear in the course of this deduction of projective Geometry from the *a priori* conceptual properties of a form of externality, are postponed to Chapter IV. The author remarks: "For the present, I wish to point out that projective Geometry is wholly *a priori*; that it deals with an object whose properties are logically deduced from its definition, not empirically discovered from data; that its definition, again, is founded on the possibility of experiencing diversity in relation, or multiplicity in unity; and that our whole science, therefore, is logically implied in, and deducible from, the possibility of such experience" (p. 146).

Projective geometry cannot distinguish between Euclidean and non-Euclidean spaces, but metrical geometry can do so by its quantitative tests. The idea of motion also appears in metrical geometry. Just as the axioms of projective geometry are deductions from the possibility of the experience of externality, the axioms of metrical geometry are deductions from the possibility of spatial measurement. The essential axioms of these geometries are equivalent and differ only slightly in form. To the axiom of the homogeneity of space corresponds that of free mobility; to the axiom of the straight line corresponds that of distance; and the axiom of dimensions is the same in both geometries. The author shows the double apriority of the axioms of metrical geometry, namely, that they are presupposed in all spatial measurement, and also that they are necessary properties of any form of externality. The argument, which cannot be repeated here, is carefully and successfully elaborated. From the investigation, it follows that "the *a priori* element (of geometry) may be defined as the axioms common to Euclidean and non-Euclidean spaces, as the axioms deducible from the conception of a form of externality, or—in metrical geometry—as the axioms required for the possibility of measurement'' (p. 177).

In the fourth chapter, the author discusses, with great modesty, some of the philosophical questions and difficulties which have arisen in the preceding portion of his essay. The first part of the chapter is taken up with a consideration of the question: "What relation can a purely logical and deductive proof, like that from the nature of a form of externality, bear to an experienced subject-matter such as space?'' Why is it possible to deduce from a mere conception, like that of a form of externality, the logical apriority of certain axioms as to experienced space? The reply is that sense-perception necessarily contains as an element some form, the conception of which is included under our form of externality.

The latter part of the chapter contains a discussion on some of the contradictions arising out of the relativity and continuity of space, which have obtruded themselves in Chapter III. The contradictions are shown to be inevitable. An attempt is made to remove them by restoring the notion of matter as that which is localized and interrelated in space. In the search for an element of space, we come upon the point, a thing in space without spatial magnitude; that is, that which is not spatial fulfils the function of a spatial element. The contradiction arises "from the attempt to deal with empty space, rather than with spatial figures, and the matter to which they necessarily refer.'' This matter is not regarded as having physical or causal properties, but as abstracted from them, and merely supplying the terms for spatial relations. "We must find, therefore, in our matter that unit of differentiation or atom, which in space we could not find:'' this atom will appear to geometry as a point. The material atom will also aid us in getting rid of the circle in the definitions of straight lines and planes. The straight line may now be defined as a spatial relation between two unextended atoms; and straight lines and planes are the true spatial units. Another contradiction is that space appears as at once relational and more than relational. On the one hand, space must be regarded as relations, and relations are necessarily indivisible. On the other hand, space appears to be infinitely divisible. In discussing this antinomy, it is essential to distinguish clearly between empty space and spatial figures. Kant argues that empty space is an intuition and not a concept, and that it must be known before spatial order becomes possible. Mr. Russell maintains, "on the contrary, that it is wholly conceptual; that space is given only as spatial order; that spatial relations, being given, appear as more than mere relations, and so become hypostatized; that when hypostatized, the whole collection of them is regarded as contained in empty space; but that this empty space itself, if it means more than the logical possibility of space relations, is an unnecessary and self-contradictory assumption" (p. 194). "Empty space is undifferentiated and homogeneous; parts of space, or spatial figures, arise only by reference to some differentiating matter, and thus belong rather to spatial order than to empty space. If empty space be the pre-condition of spatial order, we cannot expect it to be connected with spatial relations as genus with species. But empty space may nevertheless be a universal conception; it may be related to spatial order as the state to the citizens. These are not instances of the state, but are contained in it; they also, in a sense, presuppose it, for a man can only become a citizen by being related to other citizens in a state" (p. 195). Another ground for condemning empty space is found in the mathematical antinomies; for these arise only in connection with empty space, and, when only spatial order is regarded, unbounded extension and infinite divisibility both disappear. That we cannot but regard space as having thinghood and as being infinitely divisible, is explained as a psychological illusion unavoidably arising from the fact that spatial relations are immediately presented. The apparent divisibility of the relations constituting spatial order is explained by a reference to the matter described above. The conclusion is: "Space, if it is to be freed from contradictions, must be regarded exclusively as spatial order, as relations between unextended material atoms. Empty space, which arises, by an inevitable illusion, out of the spatial element in sense-perception, may be regarded, if we wish to retain it, as the bare principle of relativity, the bare logical possibility of relations between diverse things. In this sense, empty space is wholly conceptual; spatial order alone is immediately experienced" (pp. 197 f). The endeavor to overcome the contradictions in space has been made by reference to a certain matter described as having merely spatial adjectives, its non-spatial and causal properties being left out of account. The essay ends thus: "To deal with the new contradictions involved in such a notion of matter, would demand a fresh treatise, leading us, through Kinematics, into the domains of Dynamics and Physics. But to discuss the special difficulties of space is all that is possible in an essay on the Foundations of Geometry" (p. 201).

D. A. Murray.

Cornell University.